Homework 1-a, for 9/24. Math 151A, fall 2010. Deadline 10/6 but you should work on these before next monday so that the TA will have something to do.

(1) Consider the following set:

$$A := \{q \in Q | q^2 > 2, q > 0\}$$

Show that smallest element in A does not exist.

(2) Show that for any distinct rational numbers $r < r'$, there is $r'' \in \mathbb{Q}$ so that $r < r'' < r'$.

(3) Let $E = \{1/n | n \in \mathbb{Z}, n > 0\}$. Find $\inf E$ and $\sup E$ (with proof) in $\mathbb{Q}$.

Do the problems 2, 5 in page 22 of Rudin.

HW 1-a

① Consider the following set:

$$A := \{ q \in \mathbb{Q} \,/\, q^2 > 2,\ q > 0 \}.$$

Show that the smallest in A does not exist.

~~pf~~ Let $p \in \mathbb{Q}^+$. Associate to p the number

$$q = p - \frac{p^2 - 2}{p+2} = \frac{2p+2}{p+2}. \qquad (1)$$

Then $$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}. \qquad (2)$$

Then for $p \in A$ we have $p^2 > 2 \Rightarrow p^2 - 2 > 0$ and thus by (1) $q < p$ and by (2) $q^2 - 2 > 0$ $\Rightarrow q^2 > 2$, thus $q \in A$.

$\therefore \forall p \in A\ \exists q \in A$ s.t. $q < p$. $\therefore \inf A \notin A$.

② Show that for any distinct rational numbers $r < r'$, there is $r'' \in \mathbb{Q}$ so that $r < r'' < r'$.

pf/ Let $r, r' \in \mathbb{Q}$ s.t. $r < r'$.

Define $r'' = \dfrac{r + r'}{2}$.

Then $r' - r'' = r' - \dfrac{r+r'}{2} = \dfrac{2r'}{2} - \dfrac{r+r'}{2} = \dfrac{r'-r}{2} > 0$

since $r' > r$. $\therefore$ $r'' < r'$.

Now $r'' - r = \dfrac{r+r'}{2} - r = \dfrac{r+r'}{2} - \dfrac{2r}{2} = \dfrac{r'-r}{2} > 0$

$\therefore$ $r'' > r$

$\therefore$ $r < r'' < r'$

$\square$ pic.

③ Let $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Find inf E and sup E (with proof) in $\mathbb{Q}$.

pf/ <u>Claim</u> : $\sup E = 1$.

Clearly $1 \geq x \ \forall x \in E$. Since $1 \in E$ we have $\sup E = 1$.

<u>Claim</u> : $\inf E = 0$

Clearly $0 \leq x \ \forall x \in E$. Suppose $\inf E \neq 0$. Then it must be that $\inf E > 0$. Let $\inf E = \alpha$, then $\alpha = \frac{m}{n}$ for $m, n \in \mathbb{N}$. If $m > 1$ we have that $0 < \frac{1}{n} < \alpha$ and if $m = 1$ we have that $0 < \frac{1}{n+1} < \alpha$. Thus α is not the infimum of E. Thus $\inf E = 0$.

pic

Ch. 1

(2) Prove that there is no rational number whose square is 12.

pf/ Suppose $\exists p \in \mathbb{Q}$ s.t. $p^2 = 12$. Since $p \in \mathbb{Q}$ we have $p = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ $(n \neq 0)$ where m and n have no common factors.

Then $\frac{m^2}{n^2} = 12 \Rightarrow m^2 = 12n^2$. Thus m^2 is divisible by 3 and hence so is m. ($3 \mid m^2 \Rightarrow 3 \mid m$ since 3 is prime and m^2 is never prime).

$\Rightarrow m = 3a \Rightarrow 9a^2 = 12n^2 \Rightarrow 3a^2 = 4n^2 \Rightarrow 3 \mid n^2 \Rightarrow 3 \mid n$

$\divideontimes$

$\therefore \nexists p \in \mathbb{Q}$ s.t $p^2 = 12$.

$\boxed{\checkmark}$ pic

Ch. 1

5 Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf(A) = -\sup(-A)$.

pf/ By the greatest lower bound property $\inf A$ exists in $\mathbb{R}$. Let $\inf A = \alpha$.

This means that $\forall x \in A$ $\alpha \leq x$ (1) and if $y > \alpha$ (2) $(y \in \mathbb{R})$ then $y \neq \inf A$.

<u>WTS</u> $\sup(-A) = -\alpha$.

Multiplying (1) by -1 we get $-\alpha \geq -x$ $\forall x \in A$.

Thus $-\alpha \geq z$ $\forall z \in -A$. Suppose $\sup(-A) \neq -\alpha$.

Then $\exists \beta \in \mathbb{R}$ s.t. $z \leq \beta < -\alpha$ $\forall z \in -A$.

$\Rightarrow -z \geq -\beta > \alpha$ (3). Since any $x \in A$ has the form $-z$ for $z \in A$. Thus we may write (3) as

$x \geq -\beta > \alpha$ (4) which holds $\forall x \in A$. But (4) implies that $\alpha \neq \inf A$. $\divideontimes$

$\therefore \sup(-A) = -\alpha \Rightarrow \inf(A) = -\sup(-A)$. pic.

Homework 1-b, for 9/27. Math 151A, fall 2010. Deadline 10/6

(Try to start working on it before you see Edward.)
(1) Read 1.14, 1.15, 1.16. (nothing to submit.)

(2) do 8 in p22. (for now just using your knowledge on $\mathbb{C}$ from old classes: just $\{a + bi | a, b \in \mathbb{R}\}$ and products, sums, inverses are as you know.)
(3) Let $F = \{0, 1, 2\}$ be a set. We define operations $+$ and $\cdot$ by the following table:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Show that F is a field but cannot be an ordered field.

HW 1-b

① 1.14 ⓐ $x+y=x+z \Rightarrow y=z$

ⓑ $x+y=x \Rightarrow y=0$

ⓒ $x+y=0 \Rightarrow y=-x$

ⓓ $-(-x)=x$

pf/ ⓐ Suppose $x+y=x+z$. (*)

$$y \overset{(A4)}{=} 0+y \overset{(A5)}{=} (-x+x)+y \overset{(A3)}{=} -x+(x+y) \overset{(*)}{=} -x+(x+z)$$

$$\overset{(A3)}{=} (-x+x)+z \overset{(A5)}{=} 0+z \overset{(A4)}{=} z$$

$\Rightarrow y=z$

ⓑ Let $z=0$ in ⓐ. $\Rightarrow y=z=0$.

ⓒ Let $z=-x$ in ⓐ: $x+y=x+(-x)=0$

ⓐ $\Rightarrow$ $y=-x$.

ⓓ Replace x with $-x$ in ⓒ $\Rightarrow y=-(-x)$.

$\square$ pic.

1.15 (Exercise 3)

ⓐ $x \neq 0$ & $xy = xz \Rightarrow y = z$

ⓑ $x \neq 0$ & $xy = x \Rightarrow y = 1$

ⓒ $x \neq 0$ & $xy = 1 \Rightarrow y = \frac{1}{x}$

ⓓ $x \neq 0 \Rightarrow \frac{1}{\left(\frac{1}{x}\right)} = x$

Assume $xy = xz$. (☆)

pf/ ⓐ $y \overset{(M4)}{=\!=} 1y \overset{(M5)}{=\!=} \left(x \cdot \frac{1}{x}\right)y \overset{(M2)}{=\!=} \left(\frac{1}{x} \cdot x\right)y \overset{(M3)}{=\!=} \frac{1}{x}(xy)$

$\overset{(☆)}{=\!=} \frac{1}{x}(xz) \overset{(M3)}{=\!=} \left(\frac{1}{x}x\right)z \overset{(M2)}{=\!=} \left(x \cdot \frac{1}{x}\right)z \overset{(M5)}{=\!=} 1 \cdot z \overset{(M4)}{=\!=} z$

$\therefore y = z$

ⓑ Let $z = 1$ in ⓐ $\Rightarrow y = 1$.

ⓒ Let $z = \frac{1}{x}$ in ⓐ : $xy = xz = x \cdot \frac{1}{x} = 1$

ⓐ $\Rightarrow y = \frac{1}{x}$.

ⓓ Replace x with $\frac{1}{x}$ in ⓒ $\Rightarrow y = \frac{1}{\left(\frac{1}{x}\right)}$.

□ w/ pic.

1.16 $\forall x, y, z \in F$

ⓐ $0x = 0$

ⓑ $x \neq 0$ & $y \neq 0 \Rightarrow xy \neq 0$.

ⓒ $(-x)y = -(xy) = x(-y)$.

ⓓ $(-x)(-y) = xy$.

pf/ ⓐ $0x + 0x \overset{(D)}{=} (0+0)x = 0x$.

$1.14b \Rightarrow 0x = 0$.

ⓑ Assume $x \neq 0$ & $y \neq 0$ and $xy = 0$.

$$1 = \left(\frac{1}{y} \cdot y\right)\left(\frac{1}{x} \cdot x\right) = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right) xy = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right) \cdot 0 \overset{1.16a}{=} 0 \quad \times$$

$\therefore\ xy \neq 0$.

ⓒ $(-x)y + xy \overset{(D)}{=} (-x+x)y = 0y \overset{1.16a}{=} 0$

$\overset{1.14c}{\Longrightarrow} (-x)y = -(xy)$.

$x(-y) + xy = x(-y+y) = x \cdot 0 = 0$

$\overset{1.14c}{\Longrightarrow} x(-y) = -xy$

$\therefore\ (-x)y = -(xy) = x(-y)$.

ⓓ $(-x)(-y) \overset{1.16c}{=} -[x(-y)] \overset{1.16c}{=} -[-(xy)] \overset{1.14d}{=} xy$.

$\square$ pic.

② Ch. 1 #8

Prove that no order can be defined in the complex field that turns it into an ordered field.

pf/ Consider $i \in \mathbb{C}$. Since $i \neq 0$, by prop 1.18d, if $\mathbb{C}$ is an ordered field, we need $i^2 > 0$. However $i^2 = -1 < 0$. $\therefore$ $\mathbb{C}$ cannot be an ordered field.

pic.

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③ Let $F = \{0, 1, 2\}$ be a set. We define $+$ & $\cdot$ by:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	3

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

.

Show $(F, +, \cdot)$ is a field, but cannot be an ordered field.

pf/ $F \cong \mathbb{Z}_3$. B = brute force, T = trivial. Since both tables are symmetric, A2 & M2 follow.

A:

1	T
2	✓
3	B
4	T
5	✓

M:

1	T
2	✓
3	B
4	T
5	✓

$\left.\begin{array}{l} -0 = 0 \\ -1 = 2 \\ -2 = 1 \end{array}\right\} \Rightarrow A5$

$\left.\begin{array}{l} \frac{1}{(1)} = 1 \; : 1 \cdot \frac{1}{1} = 1 \cdot 1 = 1 \\ \frac{1}{(2)} = 2 \; : 2 \cdot \frac{1}{2} = 2 \cdot 2 = 1 \end{array}\right\} \Rightarrow M5$

$\therefore$ F is a field.

Suppose F could be made into an ordered field.

There are 6 possible orderings:

① $0<1<2$ ② $0<2<1$ ③ $1<0<2$ ④ $1<2<0$

⑤ $2<0<1$ ⑥ $2<1<0$.

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①: Let $x=2=z$ and $y=1$, then 1.18b implies
$xy<xz$, but $xy=2$ & $xz=1$ $\ast$

②: Let $x=2$ and $y=1$. $\frac{1}{x}=2$ & $\frac{1}{y}=1$. 1.18e implies
$0<\frac{1}{y}<\frac{1}{x}$, but $\frac{1}{x}<\frac{1}{y}$ $\ast$

③+④+⑥: Let $x=1\neq 0$. 1.18d $\Rightarrow x^2>0$, but
$x^2=1\cdot 1=1<0$ $\ast$.

⑤: Let $x=1$, $y=2$, & $z=1$. 1.17ii $\Rightarrow x+y<x+z$
But $x+y=0$ & $x+z=2<0$ $\ast$.

$\therefore$ F cannot be an ordered field.

$\square$ pic.

Homework 1-c, for 9/29. Math 151A, fall 2010. Deadline 10/6

(1) Let $A \in \mathbb{R}$ (in the sense of "cuts".) let $-A := \{p \in \mathbb{Q} | \exists r \in \mathbb{Q}_{>0}$ so that $(-p-r) \notin A\}$. Show that $-A \in \mathbb{R}$.

(2) Show that AB as defined in the class, for $A, B > 0^*$, is a cut.

HW 1-C

① Let $A \in \mathbb{R}$. Let $-A := \{p \in \mathbb{Q} \mid \exists r \in \mathbb{Q}_+ \text{ s.t. } -p-r \notin A\}$.

Show $-A \in \mathbb{R}$.

pf/ ⓘ Let $h \in \mathbb{Q}_+$. Let $p \notin A$, then $p+h \notin A$.

Claim: $-(p+h) \in -A$.

Since $-(-(p+h))-h = p+h-h = p \notin A$, $-(p+h) \in -A$.

$\therefore -A \neq \emptyset$.

Let $p \in A$. Claim: $-p \notin -A$. Let $r \in \mathbb{Q}_+$

Since $-(-p)-r = p-r \in A$ $\forall r$, $-p \notin -A$.

$\therefore -A \neq \mathbb{Q}$.

ⓘⓘ Let $p \in -A$, and let $q \in \mathbb{Q}$ be s.t. $q<p$.

$p \in -A \Rightarrow \exists r \in \mathbb{Q}_+$ s.t. $-p-r \notin A$. Since $q<p$

$\Rightarrow -q > -p \Rightarrow -q-r > -p-r \Rightarrow -q-r \notin A \Rightarrow q \in -A$.

ⓘⓘⓘ Let $p \in -A$. Then $-p-r \notin A$ for some $r \in \mathbb{Q}_+$.

Let $q := p + \frac{r}{2}$. Then choose $s = \frac{r}{2} \in \mathbb{Q}_+$, thus

$-q-s = -p-\frac{r}{2}-\frac{r}{2} = -p-r \notin A$. $\therefore q \in -A$ and $q>p$.

$\therefore -A \in \mathbb{R}$.

(2) Show that AB as defined in the class, for $A, B > 0^*$, is a cut.

pf/ Let A, B be cuts with $A, B > 0^*$.

$AB := \{ p \in \mathbb{Q} \mid p \leq ab$ for some $a \in A, b \in B$ w/ $a, b > 0 \}$

(i) It is clear that $AB \neq \emptyset$ since $0 \in AB$.
Let $c \notin \alpha$ and $d \notin \beta$, then $cd > ab$ $\forall a \in A, b \in B$.
Thus $cd \notin AB \Rightarrow AB \neq \mathbb{Q}$. ✓

(ii) Let $p \in AB$ & $q \in \mathbb{Q}$ s.t $q < p$. Then $p \leq ab$ for some $a \in A$ & $b \in B$. $q < p \Rightarrow q < p \leq ab$. $\therefore q \in AB$. ✓

(iii) Let $p \in AB$. $\Rightarrow p \leq ab$ for some $a \in A, b \in B$.
Since A & B are cuts $\exists\, a' \in A, b' \in B$ s.t. $a' > a$ & $b' > b$. Let $c = \frac{a+a'}{2}$ & $d = \frac{b+b'}{2}$. Then $p < cd \leq a'b'$
Thus $cd \in AB$. ✓

$\therefore AB$ is a cut.